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Analyticity of Schrödinger energy levels for confining potentials

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Abstract. The analyticity of the Schrödinger energy levels in the coupling constant β for a class of confining potentials of the type

$$V(r) = -\zeta/r + \alpha r + \beta r^2$$

has been studied by using Kato–Rellich perturbation theory for linear operators.

1. Introduction

The phenomenological success of the non-relativistic confinement hypothesis for heavy quarks (Appelquist and Politzer 1975) has led to considerable activity in the study of the general properties of confining potentials (Grosse and Martin 1980, Quigg and Rosner 1979, Datta and Mukherjee 1980a, b, 1982a, b). In an earlier paper (Datta and Mukherjee 1980a), we considered the class of potentials

$$V(r) = -\zeta/r + \alpha r + \beta r^2 \qquad (1.1)$$

and investigated the ζ plane analyticity of a Green function. We also studied the analyticity of the corresponding Schrödinger energy levels near $\zeta = 0$. The class of potentials (1.1) is a possible candidate for the quarkonium potential as has been indicated by the quarkonium spectroscopy (Grosse and Martin 1980). In the special case of $\beta = 0$ and $\alpha > 0$ the potential (1.1) reduces to the well known charmonium potential (Eichten *et al* 1975). Apart from its relevance in heavy quarkonium spectroscopy, the class of potentials (1.1) with $\beta = 0$ has important applications in atomic physics. The Stark effect in a hydrogen atom in one dimension is given exactly by the charmonium like potential (α being the electric field parameter). The analyticity of the energy levels $E_n(\alpha)$ in the electric field parameter α was investigated rigorously by Graffi *et al* (1979). They followed the method of Simon (1970) and Loeffel and Martin (1970), which has been developed originally to study a similar problem in connection with an anharmonic oscillator. The study leads to a better understanding of the nature of the divergence of the corresponding Rayleigh–Schrödinger energy series and also in the possibilities of its summability to yield the correct energy level.

The more general class of potentials (1.1) with $\beta > 0$ is also relevant in atomic physics. This could be interpreted as the potential seen by an electron of an atom exposed to a suitable admixture of electric and magnetic fields. Recently Rau (1979) has suggested a model to realise this type of potential. According to the model, an electron at the surface ($z=0$) of liquid helium should experience, in addition to the force due to the one-dimensional image Coulomb potential $-\zeta/z$, an electric field normal to the surface and a crossed magnetic field along the surface. The resulting potential should then assume a form similar to (1.1) i.e. $-\zeta/z+\alpha z+\beta z^2$. A systematic study of the analyticity of the energy levels for the potentials (1.1) in the confining coupling constant (β) plane, therefore, may be useful for many problems. This is what we intend to do in the present paper. It will be shown that the Rayleigh–Schrödinger energy perturbation series in β is totally divergent. The energy level $E_n(\beta)$ is, however, analytic in the first Riemann sheet of the β plane cut along the negative real axis and due to this analyticity property the divergent perturbation series is still summable in a suitable sense.

The paper is arranged as follows. In § 2 we discuss the domain problem associated with the relevant Schrödinger operator and also study the corresponding spectrum. In § 3, we prove the β analyticity of $E_n(\beta)$. We summarise the results in § 4.

2. The spectrum

Let us denote the Schrödinger operator corresponding to potentials (1.1) by

$$H(\zeta,\alpha,\beta)=-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}-\frac{\zeta}{x}+\alpha x+\beta x^2, \qquad l=0,1,2,\ldots. \tag{2.1}$$

For real ζ, α and $\beta>0$, the formal operator $H(\zeta,\alpha,\beta)$ in $L^2(0,\infty)$ is a semi-bounded self-adjoint operator with a compact resolvent on the domain

$$D(H)=\{u|u,u' \text{ absolutely continuous}|u,u'\in L^2|u(0)=0|Hu\in L^2\}. \tag{2.2}$$

The corresponding spectrum consists of discrete real eigenvalues converging only at ∞.

To study the spectrum for complex parameters we need the following two operators

$$h_1(\alpha,\beta)=-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}+\alpha x+\beta x^2, \qquad D(h_1)=D(H) \tag{2.3}$$

and

$$h_2(\zeta,\alpha)=-\frac{d^2}{dx^2}+\frac{l(l+1)}{x^2}-\frac{\zeta}{x}+\alpha x$$

$$D(h_2)=\{u|u,u' \text{ absolutely continuous}|u,u'\in L^2|u(0)=0|h_2u\in L^2\}. \tag{2.4}$$

Lemma 2.1. $D(h_2)\supset D(h_1)=D(H)$.

Proof. Let $h_0=-d^2/dx^2+l(l+1)/x^2$. Then $D(h_i)=D(h_0)\cap D(x^i)$ and $D(x^2)\subset D(x)$ where $D(x^i)$ is the domain of the maximal multiplication operator by x^i in $L^2(0,\infty)$, $i=1,2$. Hence $D(h_2)=D(h_0)\cap D(x)\supset D(h_0)\cap D(x^2)=D(h_1)$.

For β in the cut plane $|\arg\beta|<\pi$, $h_1(0,\beta)$ defines a closed operator on $D(H)$. Also for each compact set in the β cut plane there exist constants a and b such that (Simon 1970)

$$\|x^2u\|\leqslant a\|h_1(0,\beta)u\|+b\|u\|, \qquad u\in D(H). \tag{2.5}$$

Lemma 2.2. For any fixed β in the cut plane $|\arg\beta|<\pi$, $h_1(\alpha,\beta)$ is a holomorphic family in α of type A. For any fixed α in the complex plane, $h_1(\alpha,\beta)$ is a holomorphic family in β of type A as long as β belongs to $|\arg\beta|<\pi$.

Proof. The first part follows from the fact that for each ε_0, one can find δ_0 such that

$$x\leqslant\varepsilon_0x^2+\delta_0,\qquad x>0$$

and hence from (2.5) we obtain

$$\|xu\|\leqslant\varepsilon_0a\|h_1(0,\beta)u\|+(\varepsilon_0b+\delta_0)\|u\|,\qquad u\in D(H). \tag{2.6}$$

Thus x is $h_1(0,\beta)$ bounded with relative bound zero and hence the holomorphy in α follows from the standard criterion (Kato 1976). The proof of the second part follows from (2.5).

The operator family $h_2(\zeta,\alpha)$ has been studied by Graffi *et al* (1979). Making use of their results, one gets the inequality

$$\|x^{-1}u\|\leqslant c\|h_2(0,\alpha)u\|+d\|u\|,\qquad 0<c<1,\qquad u\in D(h_2) \tag{2.7}$$

where c and d are constants independent of α in compacts in the cut plane $|\arg\alpha|<\pi$.

Lemma 2.3. Let $u\in D(H)$. Then for compacts in the cut planes $|\arg\alpha|<\pi$ and $|\arg\beta|<\pi$ there exist constants a_1, b_1 such that

$$\|x^{-1}u\|\leqslant a_1\|h_1(\alpha,\beta)u\|+b_1\|u\| \tag{2.8}$$

where $0<a_1<1$.

Proof. From (2.5) and (2.6) one gets

$$\begin{aligned}\|x^2u\|&\leqslant a\|h_1(\alpha,\beta)u\|+a\|\alpha xu\|+b\|u\|\\&\leqslant a\|h_1(\alpha,\beta)u\|+a|\alpha|\varepsilon(1-\varepsilon|\alpha|)^{-1}\|h_1(\alpha,\beta)u\|+[b+a\delta|\alpha|(1-\varepsilon|\alpha|)^{-1}]\|u\|\\&\leqslant a'\|h_1(\alpha,\beta)u\|+b'\|u\|\end{aligned} \tag{2.9}$$

where

$$a'=a[1+|\alpha|\varepsilon(1-\varepsilon|\alpha|)^{-1}]$$
$$b'=[b+a\delta|\alpha|(1-\varepsilon|\alpha|)^{-1}],\qquad \varepsilon=\varepsilon_0a,\qquad \delta=\varepsilon_0a+b.$$

Next, restricting the inequality (2.7) to $D(h_1)\subset D(h_2)$ and noting the equality $h_2(0,\alpha)=h_1(\alpha,0)$ in this case, it follows that

$$\begin{aligned}\|x^{-1}u\|&\leqslant c\|h_1(\alpha,\beta)u\|+c|\beta|\|x^2u\|+d\|u\|\\&\leqslant c\|h_1(\alpha,\beta)u\|+c|\beta|a'\|h_1(\alpha,\beta)u\|+(d+c|\beta|b')\|u\|\\&\leqslant a_1\|h_1(\alpha,\beta)u\|+b_1\|u\|,\qquad u\in D(H)\end{aligned}$$

where $a_1=c(1+a'|\beta|)>0$ and $b_1=(d+c|\beta|b')>0$.

This result is, however, true even when $|\arg\alpha|=\pi$. We write $\alpha=-\alpha'$, $\alpha'>0$ in $h_1(\alpha,\beta)$ to obtain

$$h_1(\alpha,\beta)=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{l(l+1)}{x^2}+\alpha'x+\beta x^2-2\alpha'x.$$

It is now easy to see that $(\zeta/x + 2\alpha' x)$ is $h_1(\alpha', \beta)$ bounded with relative bound zero as long as $|\arg\beta| < \pi$. In fact, for $u \in D(H)$ we have

$$\begin{aligned}\|(\zeta/x + 2\alpha' x)u\| &\leq |\zeta|\|x^{-1}u\| + 2\alpha'\|xu\| \\ &\leq a_1|\zeta|\|h_1(\alpha', \beta)u\| + b_1|\zeta|\|u\| + 2\alpha'\varepsilon\|h_1(\alpha', \beta)u\| + 2\alpha'\delta\|u\| \\ &\leq (a_1|\zeta| + 2\alpha'\varepsilon)\|h_1(\alpha', \beta)u\| + (b_1|\zeta| + 2\alpha'\delta)\|u\|\end{aligned}$$

where $\varepsilon, a_1 > 0$. The middle inequality follows from (2.8) and (2.6). This completes the proof. We thus conclude the following theorem.

Theorem 2.1. Let ζ and α be finite complex numbers and β lie in a compact set in $|\arg\beta| < \pi$. Then x^{-1} is $h_1(\alpha, \beta)$ bounded with relative bound zero. The operator family $H(\zeta, \alpha, \beta)$ is a holomorphic family of type A in each parameter when the other two remain fixed in their respective domains and it has compact resolvants.

The theorem means that each eigenvalue of $H(\zeta, \alpha, \beta)$ that is non-degenerate at a point $N(\zeta_0, \alpha_0, \beta_0)$ is analytic in a neighbourhood of N. For real ζ, α and $\beta > 0$ each eigenvalue $E_n(\zeta, \alpha, \beta)$ is analytic, which follows from the self-adjointness of the corresponding $H(\zeta, \alpha, \beta)$. It also follows that the spectrum of $H(\zeta, \alpha, \beta)$ for complex parameters consists precisely of the analytically continued eigenvalues $E_n(\zeta, \alpha, \beta)$.

The above results together with the Symanzik scaling law (Simon 1970, Datta and Mukherjee 1980b)

$$E_n(\zeta, \zeta^3\alpha, 1) = \beta^{-1/2}E_n(1, \alpha, \beta), \qquad \beta = \zeta^{-4} \tag{2.10}$$

allow us to use directly the general technique developed by Simon (1970) to analyse the singularity structure of the function $E_n(\zeta, \alpha, \beta)$ in the β plane when $\alpha > 0$ and ζ is real ($\zeta > 0$ for $\alpha = 0$). It follows that for fixed ζ and α, $E_n(\zeta, \alpha, \beta)$ has an infinite number of branch points and/or natural boundaries for complex β. The point $\beta = 0$ is a limit point of these singularities and thus is an essential singularity. It is also clear from the scaling equation (2.10) that $\beta = 0$ is a fourth-order branch point of E_n. The occurrence of an essential singularity at $\beta = 0$ now tells us that the energy perturbation series $\Sigma a_n\beta^n$ in β is totally divergent. The series is, however, asymptotic to $E_n(\beta)$ uniformly in $0 < |\beta| < \varepsilon$, $|\arg\beta| < 2\pi$ (cf Graffi *et al* 1979).

3. First-sheet analyticity

Here we shall prove that the first sheet of the cut plane is free from any singularity. We shall prove this explicitly for $E_n(\zeta, \alpha, \beta)$ with fixed $\zeta > 0$, $\alpha \geq 0$. Extension of the result for negative values of ζ and α could be done trivially by analytic continuation. In the following we shall follow the arguments of Loeffel and Martin (1970). Apart from its relevance in the summability of the perturbation series, the study is of interest in itself since the potential consists of three different terms.

Lemma 3.1. When β is complex with $|\arg\beta| < \pi$, $\mathrm{Im}\, E_n/\mathrm{Im}\,\beta > 0$ for fixed real values of ζ and α. The same is true in $(-\zeta)$ with $|\arg(-\zeta)| < \pi$ and in α with $|\arg\alpha| < \pi$ for fixed real values of other parameters ($\beta > 0$).

The proof of this lemma can be obtained by following Simon (1970).

Let u_n be the wavefunction corresponding to the eigenvalue E_n. Then u_n is an entire function of the variables ζ, α and an analytic function of x in the complex plane cut along the negative real axis. It has the asymptotic behaviour

$$u_n/u_0 \to 1, \qquad u_0 = x^{-1/2} \exp\left(-\frac{\sqrt{\beta}}{2}x^2 - \frac{\alpha}{2\sqrt{\beta}}x + \frac{E_n}{2\sqrt{\beta}}\ln x\right) \tag{3.1}$$

in the sector $|\arg x| < \pi/4$. u_n is square integrable even on $|\arg x| = \pi/4$ when $\alpha > 0$.

The existence of the scaling equation (2.10) makes it sufficient to consider $E_n(1, \alpha, \beta)$, $\alpha \geqslant 0$. From the Herglotz property (lemma 3.1) we know that $E_n(1, \alpha, \beta)$ has no isolated pole or essential singularity in the β cut plane. Thus $E_n(1, \alpha, \beta)$ will be analytic in $|\arg \beta| < \pi$ if one can prove the non-existence of branch points and natural boundaries. Let us first show this in the sector $|\arg \zeta| < \pi/4$ for the function $E_n(\zeta, \zeta^3\alpha, 1)$. From equation (2.10) it follows that $E_n(\zeta, \zeta^3\alpha, 1)$ has no isolated pole or essential singularity at least in $|\arg \zeta| < \pi/4$. To show the non-existence of branch points we proceed in two steps.

Step 1. Let $\zeta > 0$. The Schrödinger equation along a ray $x = te^{i\phi}$ takes the form

$$\left(-\frac{d^2}{dt^2} + \frac{l(l+1)}{t^2} - \frac{\zeta e^{i\phi}}{t} + \zeta^3\alpha\, e^{3i\phi}t + e^{4i\phi}t^2\right)u_n = E_n\, e^{2i\phi}u_n, \qquad \alpha > 0. \tag{3.2}$$

We multiply equation (3.2) by u_n^* and then integrate partially. Taking the imaginary part we obtain

$$\operatorname{Im}(u_n' u_n^*) = \int_0^t \frac{dt'}{t'} Q(t', \zeta, \phi)|u_n|^2 \tag{3.3}$$

$$Q(t, \zeta, \phi) = t^3 \sin 4\phi + \zeta^3\alpha t^2 \sin 3\phi - E_n t \sin 2\phi - \zeta \sin \phi. \tag{3.4}$$

Since $|u_n| \to 0$ as $t \to \infty$ within $|\phi| \leqslant \pi/4$ equation (3.3) can be written as

$$\operatorname{Im}(u_n' u_n^*) = -\int_t^\infty \frac{dt'}{t'} Q(t', \zeta, \phi)|u_n|^2. \tag{3.5}$$

For $0 < \phi \leqslant \pi/4$ the cubic expression Q has only one real zero, say at $t = t_0 > 0$. This follows by an application of the Descarte rule of sign. If $t \leqslant t_0$ we take (3.3) whereas for $t > t_0$ we take (3.5). In any case the integral is of constant sign since $Q < 0$ as $t \to 0$ and $Q > 0$ as $t \to \infty$. Hence $\operatorname{Im}(u_n' u_n^*) \neq 0$ and u_n has no zero in the sector $0 < \phi \leqslant \pi/4$. The same is also true in the sector $-\pi/4 \leqslant \phi < 0$. Hence for real $\zeta > 0$, $(n-1)$ real zeros (the origin excepted) are the only zeros of u_n in the sector $|\phi| \leqslant \pi/4$.

Step 2. Let ζ be complex and $|\arg \zeta| < \pi/4$. In this case the expression (3.4) takes the form

$$Q(t, \zeta, \phi) = t^3 \sin 4\phi + |\zeta^3|\alpha t^2 \sin(3\phi + 3 \arg \zeta) - |E_n| t \sin(2\phi + \arg E_n) - |\zeta| \sin(\phi + \arg \zeta). \tag{3.6}$$

Take, for instance, $0 < \arg \zeta < \pi/4$. Then arguments similar to step 1 show that u_n has no zero in $0 \leqslant \phi < \pi/12$ and on the line $\phi = -\pi/4$. Also it follows from the asymptotic expression (3.1) that u_n has no zero as $|x| \to \infty$. For $\alpha = 0$, the number of zeros remains constant in $|\arg x| < \pi/4$. We have thus proved lemma 3.2.

Lemma 3.2. Let $u_n(x)$ be the wavefunction corresponding to the eigenvalue $E_n(\zeta, \zeta^3\alpha, 1)$, $\alpha \geq 0$.

(i) For $\zeta > 0$, $\alpha > 0$, u_n has only $(n-1)$ zeros in the sector $|\arg x| \leq \pi/4$ ($|\arg x| < \pi/4$ if $\alpha = 0$).

(ii) For complex ζ with $|\arg \zeta| < \pi/4$, u_n has no zero (a) in $0 \leq \arg x < \pi/12$ and on the line $\phi = -\pi/4$ when $0 < \arg \zeta < \pi/4$, (b) in $-\pi/12 < \arg x \leq 0$ and on the line $\phi = \pi/4$ when $-\pi/4 < \arg \zeta < 0$.

(iii) For $\alpha = 0$, u_n has $(n-1)$ zeros in $|\arg x| < \pi/4$ for any ζ with $|\arg \zeta| < \pi/4$. In any case, $u_n(x)$ has no zero at infinity.

Let us now assume that $\zeta = \zeta_0$ is a branch point of E_n. If we vary ζ continuously starting from a point on the real ζ axis, along a path that encircles and returns back to the initial point, $(n-1)$ zeros of $u_n(x)$ (which are continuous functions of ζ) also vary continuously remaining within the sector $|\arg x| < \pi/4$. Moreover, no zero can enter the sector from infinity or from the exterior of the sector. Thus as ζ returns to the initial real point the number of zeros cannot change and we get back the same energy level E_n. This is contrary to the assumption that ζ_0 is a branch point and hence the non-existence of branch points in $|\arg \zeta| < \pi/4$ is proved.

In the previous paragraph we have tacitly assumed that as ζ varies continuously E_n also does so and hence it does not become infinitely large. To prove that E_n remains bounded we consider the integral representation of the corresponding Schrödinger equation (de Alfaro and Regge 1965).

$$u_n(x) = \sqrt{x} J_\nu(\sqrt{E_n} x) + \tfrac{1}{2}\pi(-1)^l \int_0^x [J_\nu(\sqrt{E_n} x) J_{-\nu}(\sqrt{E_n} x') - J_\nu(\sqrt{E_n} x') J_{-\nu}(\sqrt{E_n} x)]$$

$$\times \sqrt{x x'} \left(-\frac{\zeta}{x'} + \zeta^3 \alpha x' + x'^2 \right) u_n(x')\, dx', \qquad \nu = l + \tfrac{1}{2}. \tag{3.7}$$

For $|x| < R$, $R > 0$ it can be shown that

$$|u_n(x) - \sqrt{x} J_\nu(\sqrt{E_n} x)| (e^{\mathrm{Im}\sqrt{E_n} x})^{-1} \to 0 \qquad \text{as } |E_n| \to \infty. \tag{3.8}$$

It follows that in the limit $|E_n| \to \infty$ we have

$$|\sqrt{x} J_\nu(\sqrt{E_n} x)| > |u_n(x) - \sqrt{x} J_\nu(\sqrt{E_n} x)| \tag{3.9}$$

for finite $|x|$. Let $v(x) = \sqrt{x} J_\nu(\sqrt{E_n} x)$. Since both $u_n(x)$ and $v(x)$ are analytic functions of x in the cut plane, we can now apply Rouche's theorem for a suitable finite region in $|\arg x| < \pi/4$. By virtue of the relation (3.9), $u_n(x)$ and $v(x)$ should have the same number of zeros in the finite region. Since $|E_n|$ is arbitrarily large, this would imply that $u_n(x)$ should have a large number of zeros, contrary to lemma 3.2. Hence E_n must remain bounded except only along the real axis. This completes the proof of the analyticity of E_n in the sector $|\arg \zeta| < \pi/4$. Applying the scaling equation (2.10) we obtain theorem 3.1.

Theorem 3.1. The energy level E_n is analytic in the cut plane $|\arg \beta| < \pi$.

4. Conclusion

In an earlier paper (Datta and Mukherjee 1980a) we initiated the study of the coupling constant analyticity of the Schrödinger energy levels E_n for the three-term potential

(1.1). Analytic continued fraction theory was used to study the ζ-plane analyticity of a J-fraction representation of a Green function and it was shown that the Green function was meromorphic in ζ for fixed real E, α and $\beta > 0$. The meromorphy of the Green function helped us, in turn, to prove the weak-coupling analyticity of $E_n(\zeta)$ near $\zeta = 0$. For $\alpha = 0$, a square-root branch point at $\zeta = 0$ was encountered. This, however, is not correct. In this paper we have extended our investigations using a more general approach. It follows from the self-adjointness of the corresponding Schrödinger operator for real ζ, α and $\beta > 0$ that the energy levels E_n should be analytic for real values of these parameters. Thus the square-root branch point in ζ for $\alpha = 0$, as found in our earlier derivation, is spurious. Details of the nature of the eigenvalue condition in the form of an infinite-dimensional Hill's determinant equation will be reported elsewhere. For fixed $\alpha \geqslant 0$, $\zeta > 0$, it is further shown that $E_n(\beta)$ is analytic in $|\arg \beta| < \pi$. The point $\beta = 0$ is an essential singularity which renders the corresponding Rayleigh–Schrödinger series $\Sigma a_n \beta^n$ totally divergent. Arguments similar to Graffi *et al* (1979) could be easily applied to show that the series is asymptotic to $E_n(\beta)$ uniformly in $0 < |\beta| < \varepsilon$, $|\arg \beta| < 2\pi$ and the coefficients a_n are bounded by $(2n)!$. The bounds of a_n together with the analyticity property lead us to conclude that the divergent series is (generalised) Borel summable (Graffi *et al* 1970) to E_n (β) in $|\arg \beta| < \pi$. Moreover, combining together (1) the Herglotz property, (2) the first sheet analyticity, (3) the asymptotic nature of the series and the bounds of the a_n and (4) the asymptotic behaviour $E_n(1, \alpha, \beta) \xrightarrow[\beta \to \infty]{} \beta^{1/2} E_n(0, 0, 1)$ which follows from the scaling relation (2.10), we see that the Stieltzes continued fraction corresponding to the perturbation series exists and converges to $E_n(\beta)$ uniformly on compacts of $|\arg \beta| < \pi$.

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